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Motion of charged particles in a helically perturbed magnetic field

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Abstract. The motion of a charged particle in a magnetic field composed of a uniform axial component and a spatially rotating transverse component is analysed. Solutions of the nonlinear equations of motion are obtained which describe an oscillatory transfer of kinetic energy between axial motion and rotational motion in the transverse plane. A simple physical description of the interaction in terms of impulsive Lorentz forces is given which has formal similarity to the Huygens' principle in physical optics.

1. Introduction

Magnetic fields derived from the superposition of a uniform axial component and a transverse component rotating in direction with displacement along the axis occur in a number of physical situations. Such fields were employed by Spitzer (1958) in the Stellarator, by Wingerson (1961) and others (Dreicer *et al.* 1962 and Karr *et al.* 1963) for the trapping of plasma particles between magnetic mirrors, and by a number of workers (Hirshfield and Wachtel 1964, Bott 1965, Robinson 1969 and Kulke—private communication) interested in the pumping of free electrons into eigenstates of the electron cyclotron maser. They play a prime role in the cyclotron resonance detector–spectrometer system recently discussed for the far-infrared region of the spectrum (Robinson—to be published). Theoretical studies of particle–field interactions have been carried out by Wingerson *et al.* (1964) for the case of bifilar and quadrifilar current-carrying helices with varying pitch chosen to reduce the probability of particle escape through the loss cone of a magnetic mirror. The procedure used by Wingerson *et al.* assumes resonance of the helical field and the helical particle orbit throughout the interaction and prescribes suitable velocity and field variations along the axial direction. Lidsky (1964) has discussed oscillations of non-resonant particles about the synchronous (resonant) particle orbit.

Under some conditions synchronous motion of the particle and the helical field cannot occur. In particular, this is so for the helix of constant pitch used in cyclotron maser work where the transfer of motion from the axial direction to the transverse plane necessarily involves the loss of any initial synchronism. It is our purpose in the present paper to analyse the transfer of energy between the axial or z direction and the transverse plane in a helix of constant pitch.

2. The magnetic field

Take the uniform component of magnetic induction field to be in the z direction and denote it B_z. The transverse component is derived from two (or four) helical windings each with pitch P and displaced relative to one another along the z axis by a distance $P/2$. It can be shown (Poritsky 1959) that the magnetic scalar potential of such an arrangement can, in the limit of infinite helix length, be written as the sum of a series of n spatial harmonics of the variable nkz, where $k = 2\pi/P$ and $n = 1, 2, \ldots$, each with a variation in the transverse plane given by the modified Bessel function $I_n(nkr)$, where $r = (x^2+y^2)^{1/2}$ is the radial distance from the axis of the helices. If the currents in the two (or four) helices are equal but oppositely directed their contributions on axis to the field in the z direction cancel. Furthermore, as Wingerson *et al.* (1964) point out, the spatial harmonics with n even also cancel, while those with $n = 3, 5$, etc. are small in amplitude and may be neglected in comparison with the $n = 1$ harmonic. Following these workers we can therefore write the cartesian components of the transverse helical field as

$$B_x = b\{I_0(kr)+I_2(kr)\}\sin kz$$

$$B_y = b\{I_0(kr)+I_2(kr)\}\cos kz.$$

We shall restrict the region of interaction to the near vicinity of the axis where $kr \ll 1$ so that we may take

$$B_x = b \sin kz \tag{1}$$

$$B_y = b \cos kz \tag{2}$$

where b is the constant magnitude of the transverse magnetic induction field. Now, in practice, the helices are, of course, not infinitely long and it is necessary to assess the effect of finite length. To this end computer calculations based on Ampère's law have been carried out (1969) to determine the range of the dimensions of a bifilar arrangement for which the above expressions are applicable. It has been found that provided the pitch and length of the helix are not too short the field over most of its length is given by expressions (1) and (2) and it falls off rapidly beyond the ends.

3. Solution of the equations of motion

Consider a particle of charge q and mass m travelling in the $+z$ direction with velocity $\dot{z}(0)$ parallel to the uniform field B_z. At time $t = 0$ and position $z = 0$ it encounters the helical field directed in the x–y plane and rotating in direction as z increases in accordance with expressions (1) and (2). At time t the particle emerges from the end of the helix where the transverse field terminates. During the interaction Lorentz forces act on the charge to translate some of the z-directed momentum into momentum in the x–y plane and give to the particle a helical trajectory.

The equations of charge motion are:

$$\ddot{x} = -\omega_b \dot{z} \cos kz + \omega_c \dot{y} \tag{3}$$

$$\ddot{y} = \omega_b \dot{z} \sin kz - \omega_c \dot{x} \tag{4}$$

$$\ddot{z} = -\omega_b \dot{y} \sin kz + \omega_b \dot{x} \cos kz \tag{5}$$

where

$$\omega_c = \frac{qB_z}{m} \quad \text{and} \quad \omega_b = \frac{qb}{m} \tag{6}$$

are cyclotron frequencies associated with the two orthogonal components of the magnetic induction field. The equations contain no source of energy other than that at injection at time $t = 0$; they include the conservation principle

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \dot{z}^2(0) \tag{7}$$

where $\dot{z}^2(0)$ is a constant.

We proceed to solve these nonlinear equations subject to the initial conditions

$$\dot{x} = \dot{y} = 0 \text{ at } z = 0 \text{ and } t = 0. \tag{8}$$

Set

$$a = \frac{\omega_b}{k}, \qquad \beta = \frac{1}{2}\frac{\omega_b{}^2}{\omega_c{}^2}, \qquad Z = kz \tag{9}$$

and define F and G by

$$\dot{x} = a(F - \sin Z) \tag{10}$$

$$\dot{y} = a(G - \cos Z). \tag{11}$$

Also set

$$\tau = \omega_c t. \tag{12}$$

Then if $\dot{F}$, $\dot{G}$, $\dot{Z}$ denote differentiation with respect to τ, equations (3), (4) and (5) take the form

$$\dot{F} = G - \cos Z \tag{13}$$

$$\dot{G} = -F + \sin Z \tag{14}$$

$$\ddot{Z} = 2\beta(F \cos Z - G \sin Z). \tag{15}$$

Substitute

$$\cos Z = G - \dot{F}, \qquad \sin Z = \dot{G} + F$$

into the last equation:

$$\ddot{Z} + 2\beta(G\dot{G} + \dot{F}F) = 0$$

$$\dot{Z} + \beta(G^2 + F^2) = \lambda$$

where λ is a constant of integration.

Setting

$$H = G^2 + F^2 \tag{16}$$

we have

$$\dot{Z} = \lambda - \beta H. \tag{17}$$

Now regard $F = F(Z)$, $G = G(Z)$ as functions of Z, and denote by F' and G' derivatives with respect to Z. Then

$$\dot{F} = F'\dot{Z} = F'(\lambda - \beta H)$$

$$\dot{G} = G'\dot{Z} = G'(\lambda - \beta H)$$

and from (13) and (14)

$$F'(\lambda - \beta H) = G - \cos Z \tag{18}$$

$$G'(\lambda - \beta H) = -F + \sin Z. \tag{19}$$

Equations (18) and (19) give

$$FF' + GG' = F' \sin Z + G' \cos Z$$

and hence, by (16),

$$\tfrac{1}{2}H' = F' \sin Z + G' \cos Z. \tag{20}$$

Multiply (18) by F, (19) by G and add:

$$\tfrac{1}{2}H'(\lambda - \beta H) = G \sin Z - F \cos Z. \tag{21}$$

Now

$$\begin{aligned}(F \sin Z + G \cos Z)' &= (F' \sin Z + G' \cos Z) + (F \cos Z - G \sin Z) \\ &= F' \sin Z + G' \cos Z - \tfrac{1}{2}H'(\lambda - \beta H) \qquad \text{(by (21))} \\ &= \tfrac{1}{2}H'(1 - \lambda) + \tfrac{1}{2}\beta H H' \qquad \text{(by (20))}\end{aligned}$$

and by integration

$$F\sin Z+G\cos Z = -\tfrac{1}{2}(\lambda-1)H+\tfrac{1}{4}\beta H^2+\tfrac{1}{2}\mu \tag{22}$$

where μ is a constant of integration.

From (21) and (22)

$$F = -\tfrac{1}{2}(\lambda-1)H\sin Z+\tfrac{1}{4}\beta H^2\sin Z+\tfrac{1}{2}\mu\sin Z-\tfrac{1}{2}H'(\lambda-\beta H)\cos Z \tag{23}$$

$$G = -\tfrac{1}{2}(\lambda-1)H\cos Z+\tfrac{1}{4}\beta H^2\cos Z+\tfrac{1}{2}\mu\cos Z+\tfrac{1}{2}H'(\lambda-\beta H)\sin Z. \tag{24}$$

Substituting into (16) gives

$$H = \{-\tfrac{1}{2}(\lambda-1)H+\tfrac{1}{4}\beta H^2+\tfrac{1}{2}\mu\}^2+\tfrac{1}{4}H'^2(\lambda-\beta H)^2$$

or

$$(\lambda-\beta H)\frac{\mathrm{d}H}{\mathrm{d}Z} = [4H-\{\mu-(\lambda-1)H+\tfrac{1}{2}\beta H^2\}^2]^{1/2}. \tag{25}$$

The initial conditions (8) when inserted in (10), (11) and (16) give $F(0) = 0$, $G(0) = 1$, $H(0) = 1$, and, from (13), (14) and (20), $\dot{F}(0) = \dot{G}(0) = H'(0) = 0$. Thus from (22)

$$\mu = \lambda+1-\tfrac{1}{2}\beta. \tag{26}$$

With this value of μ equations (17) and (25) give

$$(\lambda-\beta H)H' = H'\dot{Z} = \dot{H} = [4H-\{\lambda+1-(\lambda-1)H+\tfrac{1}{2}\beta(H^2-1)\}^2]^{1/2}. \tag{27}$$

Hence

$$\tau = \int_1^H \frac{\mathrm{d}h}{[4h-\{\lambda+1-(\lambda-1)h+\tfrac{1}{2}\beta(h^2-1)\}^2]^{1/2}} \tag{28}$$

which gives H as an elliptic function of τ.

3.1. *Synchronous injection*

We suppose now that the particle enters the helical field with the synchronous velocity. That is,

$$\dot{z}(0) = \frac{\mathrm{d}z(0)}{\mathrm{d}t} = \frac{\omega_c}{k}$$

or, by (9) and (12),

$$\dot{Z}(0) = \frac{k}{\omega_c}\frac{\mathrm{d}z(0)}{\mathrm{d}t} = 1. \tag{29}$$

Then, by (17)

$$\lambda = 1+\beta. \tag{30}$$

We note that β is a small quantity: $B_z \sim 100$ gauss, the helical field b will, in general, be of the order of or less than 1 gauss and $\beta \lesssim 10^{-4}$.

Substitution of (30) into (28) gives

$$= \int_0^{H-1} \frac{\mathrm{d}h}{(4h-2\beta h^2-\tfrac{1}{4}\beta^2h^4)^{1/2}}$$

$$= \frac{1}{2\alpha}\int_0^{\alpha^2(H-1)} \frac{\mathrm{d}p}{(p-2\alpha p^2-p^4)^{1/2}} \qquad \text{(where } h = p/\alpha^2\text{)}$$

and

$$\alpha = (\beta/4)^{1/3}. \tag{31}$$

Thus

$$H = 1+\alpha^{-2}\,T(\alpha\tau) \tag{32}$$

where $T(v)$ is an elliptic function given by

$$v = \tfrac{1}{2}\int_0^{T(v)} \frac{\mathrm{d}p}{(p-2\alpha p^2-p^4)^{1/2}}$$

$$\simeq \tfrac{1}{2}\int_0^{T(v)} \frac{\mathrm{d}p}{[p(1-\frac{2}{3}\alpha-p)\{1+\frac{2}{3}\alpha+(1-\frac{2}{3}\alpha)p+p^2\}]^{1/2}}.$$

This can be reduced to the standard form

$$v = \frac{1}{2(3)^{1/4}}\,\mathrm{cn}^{-1}\left\{\frac{-(\sqrt{3}+1+\frac{2}{3}\alpha)T(v)+1}{(\sqrt{3}-1-\frac{2}{3}\alpha)T(v)+1},\,K\right\} \tag{33}$$

where

$$K = \tfrac{1}{2}(2-\sqrt{3})^{1/2} = \sin 15°. \tag{34}$$

Thus in terms of elliptical functions we have

$$T(v) = \frac{1-\mathrm{cn}\{2(3)^{1/4}v,\,K\}}{(\sqrt{3}+1+\frac{2}{3}\alpha)+(\sqrt{3}-1-\frac{2}{3}\alpha)\,\mathrm{cn}\{2(3)^{1/4}v,\,K\}}. \tag{35}$$

In particular, from (33), the period of $T(v)$ is 2γ where

$$\gamma = 3^{-1/4}\,\mathrm{cn}^{-1}(0,K) = 3^{-1/4}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1-\sin^2\frac{1}{12}\pi\sin^2\theta)^{1/2}}. \tag{36}$$

From tables of elliptic functions (Jahnke and Emde 1945) we find

$$\gamma = 1{\cdot}2143. \tag{37}$$

These results lead to the velocity of the particle. Substituting from (30), (31) and (32) into (17) we get

$$\dot{Z} = 1-\beta\alpha^{-2}T(\alpha\tau)$$

or, in the same approximation as before

$$\dot{Z} = 1-2(2\beta)^{1/3}T(\alpha\tau).$$

Hence $\dot{Z}^2$ oscillates below 1 with period

$$\frac{2\gamma}{a} = 4{\cdot}857\left(\frac{\omega_{\mathrm{c}}}{\omega_{\mathrm{b}}}\right)^{2/3} \tag{38}$$

and amplitude

$$\frac{4(2\beta)^{1/3}}{1+\frac{2}{3}\alpha} = \frac{4(\omega_{\mathrm{b}}/\omega_{\mathrm{c}})^{2/3}}{1+\frac{1}{3}(\omega_{\mathrm{b}}/\omega_{\mathrm{c}})^{2/3}}. \tag{39}$$

From (23), (24), (25), (26), (31) and (32) we obtain with suitable approximation

$$F = \alpha^{-1}T^2(\alpha\tau)\sin\tau-\alpha^{-1}\{T(\alpha\tau)-2\alpha T^2(\alpha\tau)-T^4(\alpha\tau)\}^{1/2}\cos\tau \tag{40}$$

$$G = \alpha^{-1}T^2(\alpha\tau)\cos\tau+\alpha^{-1}\{T(\alpha\tau)-2\alpha T^2(\alpha\tau)-T^4(\alpha\tau)\}^{1/2}\sin\tau \tag{41}$$

where $\alpha = \frac{1}{2}(\omega_{\mathrm{b}}/\omega_{\mathrm{c}})^{2/3}$. Substitution into (10) and (11) now gives the cartesian components of particle velocity

$$\dot{x} = \frac{2}{k}\,\omega_c\left(\frac{\omega_b}{\omega_c}\right)^{1/3}\Big[T^2(\tfrac{1}{2}\Omega t)\sin\omega_c t$$

$$-\left\{T(\tfrac{1}{2}\Omega t)-\left(\frac{\omega_b}{\omega_c}\right)^{2/3}T^2(\tfrac{1}{2}\Omega t)-T^4(\tfrac{1}{2}\Omega t)\right\}^{1/2}\cos\omega_c t\Big] \tag{42}$$

$$\dot{y} = \frac{2}{k}\,\omega_c\left(\frac{\omega_b}{\omega_c}\right)^{1/3}\Big[T^2(\tfrac{1}{2}\Omega t)\cos\omega_c t$$

$$+\left\{T(\tfrac{1}{2}\Omega t)-\left(\frac{\omega_b}{\omega_c}\right)^{2/3}T^2(\tfrac{1}{2}\Omega t)-T^4(\tfrac{1}{2}\Omega t)\right\}^{1/2}\sin\omega_c t\Big] \tag{43}$$

where Ω is defined below.

Thus

$$\dot{x}^2+\dot{y}^2 \simeq \frac{4}{k^2}\,\omega_c^{\,2}\left(\frac{\omega_b}{\omega_c}\right)^{2/3}T(\tfrac{1}{2}\Omega t)-\frac{4}{k^2}\,\Omega^2T^2(\tfrac{1}{2}\Omega t) \tag{44}$$

and

$$\dot{z}^2 = \frac{\omega_c^{\,2}}{k^2}\,\dot{Z}^2 \simeq \frac{\omega_c^{\,2}}{k^2}-\frac{4}{k^2}\,\omega_c^{\,2}\left(\frac{\omega_b}{\omega_c}\right)^{2/3}T(\tfrac{1}{2}\Omega t). \tag{45}$$

In equations (42) to (45) we have introduced the notation

$$\Omega = (\omega_c\omega_b^{\,2})^{1/3}. \tag{46}$$

The square of the axial velocity, $\dot{z}^2$, oscillates with period

$$\frac{4\gamma}{\Omega} = \frac{4\cdot857}{\Omega}$$

and magnitude

$$\frac{4\omega_c^{\,2}(\omega_b/\omega_c)^{2/3}}{k^2\{1+\frac{1}{3}(\omega_b/\omega_c)^{2/3}\}}.$$

Let us take as an example an electron in a field $B_z = 50$ gauss entering a transverse helical field $b = 0\cdot1$ gauss with the synchronous velocity ω_c/k at time $t = 0$. We have $\omega_c = 2\pi\times1\cdot4\times10^8\ \mathrm{s}^{-1}$, $\omega_b = 2\pi\times2\cdot8\times10^5\ \mathrm{s}^{-1}$, $\alpha = \frac{1}{2}(1/500)^{2/3}$ and we can calculate values of $T(\frac{1}{2}\Omega t)$ from (35). The elliptic cosine $\mathrm{cn}\,u = \mathrm{cn}\,(u, k)$, $K = \sin\alpha'$ is defined by $\mathrm{cn}\,u = \cos\phi$ where

$$u = F(\phi|\alpha') = \int_0^\phi (1-\sin^2\alpha'\sin^2\theta)\,\mathrm{d}\theta.$$

In the present case $\alpha' = 15°$, and values of $F(\phi|\alpha')$—elliptic integrals of the first kind—can be found from tables (Jahnke and Emde 1945). From (44) we can then calculate $(k^2/\omega_c^{\,2})\,(\dot{x}^2+\dot{y}^2)$, the fraction of the injected energy translated from the axial direction to motion in the transverse plane. The result, as plotted in figure 1(*a*), shows the oscillatory transfer of a fraction of the energy between axial and transverse motion. Some 6% of the energy oscillates with a period $0\cdot347\ \mu$s. Figure 1(*b*) is a similar plot for $B_z = 50$ gauss and $b = 1\cdot0$ gauss. This larger transverse field transfers 29% of the energy with a period of $0\cdot0748\ \mu$s.

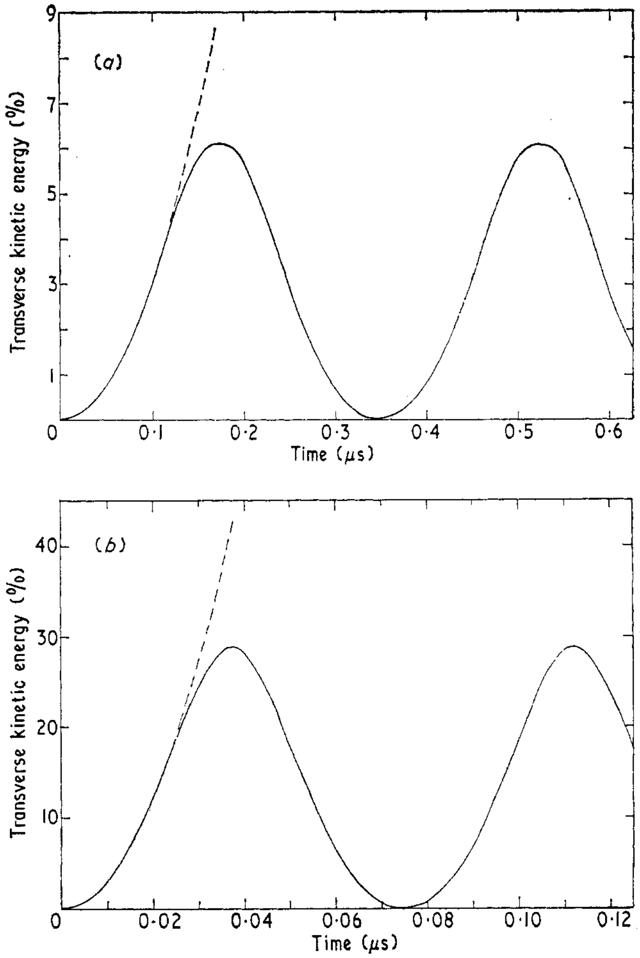

Figure 1. Percentage of the kinetic energy of an electron translated into circular motion by a helical magnetic field as a function of time. Initially the particle has the synchronous velocity and is moving parallel to a uniform field of 50 gauss. In (*a*) the transverse helical field is 0·1 gauss, and in (*b*) it is 1·0 gauss. The curves show an oscillatory transfer of a fraction of the energy between linear and circular motion. The broken curves show the parabolic approximation to the energy growth at early times discussed in § 4.

3.2. *Asynchronous injection*

For injection velocities away from ω_c/k (that is, λ is not near unity) we integrate (28) by assuming first that $\beta = 0$. The result is

$$H_{(\beta=0)} = \frac{\lambda^2+1}{(\lambda-1)^2} - \frac{2\lambda}{(\lambda-1)^2}\cos(\lambda-1)\tau. \tag{47}$$

Hence, by (23) and (24),

$$F_{(\beta=0)} = -\frac{1}{\lambda-1}\sin\lambda\tau + \frac{\lambda}{\lambda-1}\sin\tau \tag{48}$$

$$G_{(\beta=0)} = -\frac{1}{\lambda-1}\cos\lambda\tau + \frac{\lambda}{\lambda-1}\cos\tau. \tag{49}$$

For β small but not zero, equations (10) and (11) give

$$\dot{x} = \frac{\omega_b}{k}\frac{\lambda}{\lambda-1}(\sin\omega_c t - \sin\lambda\omega_c t) \tag{50}$$

$$\dot{y} = \frac{\omega_b}{k}\frac{\lambda}{\lambda-1}(\cos\omega_c t - \cos\lambda\omega_c t) \tag{51}$$

$$\dot{x}^2+\dot{y}^2 = \frac{2\lambda^2}{(\lambda-1)^2}\frac{\omega_b{}^2}{k^2}\{1-\cos(\lambda-1)\omega_c t\}. \tag{52}$$

The velocity along the z axis can be obtained from (17). If β is small so that β^2 can be neglected we obtain, apart from a small correction term,

$$\dot{z}^2 = \frac{\omega_c{}^2}{k^2}\lambda^2 + \frac{2\omega_b{}^2}{k^2}\frac{\lambda^2}{(\lambda-1)^2}\{\cos(\lambda-1)\omega_c t - 1\}. \tag{53}$$

In reducing (17) to (53), terms are neglected which require $(\lambda-1)^2 \gg \beta$. That is, by (17), results (52) and (53) apply to particles with injection velocities satisfying the condition

$$\left|\frac{k}{\omega_c}\dot{z} - 1\right| \gg \beta^{1/2}. \tag{54}$$

Although we have chosen to derive the results for asynchronous injection on the basis of our previous development, we point out that equations (3) to (8) can be solved directly by perturbation methods to yield equations (50) to (54) as solutions to lowest order in β. However, perturbation methods cannot be used in the case of synchronous injection discussed in § 3.1.

4. Physical description of the interaction

The complexity of the foregoing analysis of the equations of motion raises the need for a simple physical description of the interaction. We now take up this matter. We show that by considering the charged particle in cyclotron motion under the influence of a succession of elemental impulsive Lorentz forces in the transverse plane the behaviour can be readily understood, and approximate expressions resembling some of the foregoing solutions can be derived.

Consider the total time t of interaction to be divided into a succession of sub-intervals each of length dt. In each interval dt the momentum translated to the transverse plane will be

$$dP = q\dot{z}b\,dt.$$

Let us regard $\dot{z}$ as essentially constant and much greater than $\dot{x}$ and $\dot{y}$. Under these conditions the helical motion of the synchronous particle can stay in phase with the Lorentz forces rotating with the helical field $\boldsymbol{b}$, and the net momentum translation will be $q\dot{z}bt$. The energy of rotational motion acquired is

$$\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{q^2\dot{z}^2b^2t^2}{2m}.$$

The synchronous condition is $\omega_c t - k\dot{z}t = 0$. This parabolic growth of transverse energy is represented by the broken curves in figure 1.

In the helix of constant pitch a phase shift between successive Lorentz impulses must occur and the motion of the particle will be closely related to this phase shift. Let us again distinguish two cases: that of synchronous injection where the phase shift of the particle cyclotron motion with respect to the rotation of the field $\boldsymbol{b}$ develops as a consequence of the decrease of $\dot{z}$ by Lorentz forces, and the asynchronous case where the phase difference is due almost entirely to the initial asynchronism. In the former case one can see qualitatively the origin of the oscillatory energy exchange. In the early stages of the interaction the succession of impulses act constructively to 'wind up' the particle with an approximately parabolic growth of energy with time. Later, with the development of the shift of phase, they become so directed as to retard the circular motion and restore the energy to the axial direction.

The decrease of $\dot{z}$ below the synchronous velocity ω_c/k reduces the effective rotational frequency of the field $\boldsymbol{b}$ below ω_c and the electron develops a phase lead ϕ given by

$$\phi = 2\Omega \int_0^t T(\tfrac{1}{2}\Omega t)\,\mathrm{d}t. \tag{55}$$

Figure 2 shows the variation of ϕ with time for the values of field chosen in figure 1(a). Comparison of these two figures shows that energy is translated into circular motion

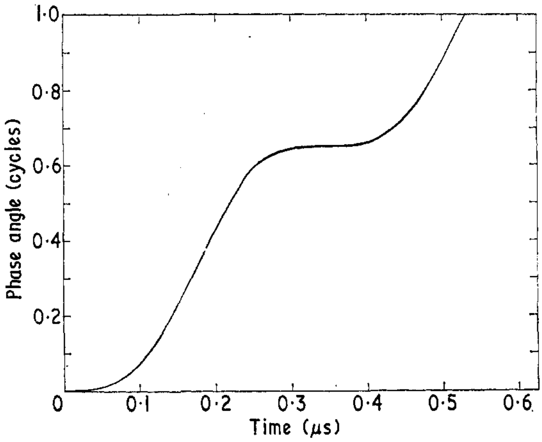

Figure 2. The phase angle developed between Lorentz forces on an electron as a function of time. The conditions correspond to those of figure 1(a). The helical motions of the electron and the transverse magnetic field are initially synchronous, but a phase slippage occurs as energy is transferred between linear and circular motion.

until the phase has changed by about $\frac{1}{3}\pi$, and it is restored to the axial direction during the next third of a cycle. The phase increases continuously except when the motions become momentarily synchronous when the momentum is totally z-directed. Similar phase shifts are associated with the oscillations of energy for other values of the fields B_z and b.

In the case of asynchronous injection the phase difference between successive Lorentz impulses will be

$$\delta = (\omega_c - k\dot{z})\,\mathrm{d}t.$$

Let us regard $\dot{z}$ as essentially constant and add the impulses vectorially as indicated in figure 3 to find the resultant translated momentum. It is proportional to the ratio of the cord to the arc of a circle subtending an angle Γ at the centre, and has the value

$$P = q\dot{z}bt\frac{\sin\Gamma/2}{\Gamma/2}$$

where

$$\Gamma = (\omega_c - k\dot{z})t$$

is the total phase shift developed during the interaction. Thus

$$\dot{x}^2+\dot{y}^2 = \frac{P^2}{m^2} = \frac{q^2\dot{z}^2b^2t^2}{m^2}\left\{\frac{\sin(\omega_c-k\dot{z})t/2}{(\omega_c-k\dot{z})t/2}\right\}^2 \tag{56}$$

$$= \frac{2\omega_b{}^2\dot{z}^2}{(\omega_c-k\dot{z})^2}\{1-\cos(\omega_c-k\dot{z})t\} \tag{57}$$

which is virtually identical with equation (52). We note that (57) is the solution of

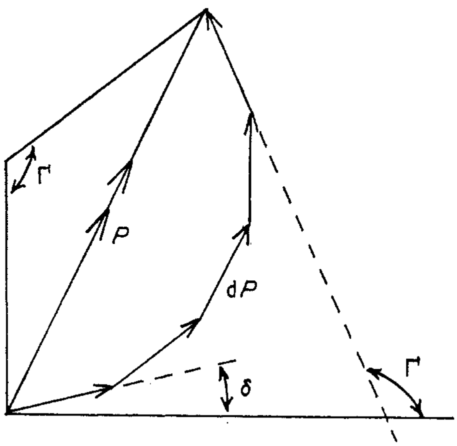

Figure 3. Construction of the net momentum P imparted by a succession of elemental impulses dP. The phase shift between successive impulses is δ and that between the first and last is Γ.

the linear equations obtained when $\dot{z}$ is approximated as a constant in (3) and (4). One expects this to be a reasonable approximation provided the phase change associated with the small decrease in $\dot{z}$ indicated by (57) is much smaller than Γ. That is, provided

$$\dot{z}^2 \gg \frac{\omega_b{}^2\dot{z}^2}{(\omega_c-k\dot{z})^2}$$

or

$$\left|1-\frac{k\dot{z}}{\omega_c}\right| \gg \beta^{1/2} \tag{58}$$

which is the same condition as (54).

Equation (57) and the physical argument leading to it, and also the accurate equation (52), apply to the velocities both below and above the synchronous velocity. In (56) and (52) the particle cyclotron motion slips with respect to the helical field when $\dot{z} > \omega_c/k$, while in the case $\dot{z} < \omega_c/k$ it gains in phase relative to $\boldsymbol{b}$. In the frame of reference moving along the z axis with the velocity of the particle the $\boldsymbol{b}$ field becomes static in the limiting case of zero drift velocity in the z direction and reverses its direction for particle motion in the $-z$ direction. Under these conditions the particle and field helical motions are 'crossed'. The previous picture still applies but the phase shift must now be the sum of $\omega_c t$ and $k\dot{z}t$. Hence, for reversal of direction (or a change of particle sign) we have

$$\dot{x}^2+\dot{y}^2 = \frac{2{\omega_b}^2\dot{z}^2}{(\omega_c+k\dot{z})^2}\{1-\cos{(\omega_c+k\dot{z})t}\}. \tag{59}$$

The particle oscillates rapidly with very small amplitude.

A useful analogy exists between the present interaction and effects in physical optics. It should be noted that the simple physical argument used here is similar to that used in optical diffraction and interference phenomena. For example, in Fourier transform spectroscopy the incident wave amplitude is first divided into two beams which are recombined with a succession of phase shifts imposed between the beams. The resulting superposition, or interferogram as it is called, gives an output intensity which is modulated by a sinc squared function of the phase shift similar in form to (55). Again, in the theory of single-slit diffraction the intensity results from the superposition of amplitudes of Huygens' wavelets originating from elemental subdivisions of the aperture, with a succession of phase shifts determined by the positions of these elemental 'sources'. The sinc squared function in this case is well known. Going further, by analogy, one might expect the transverse energy of a charged particle after passage through several bifilar helices to be analogous to that of the intensity due to a multiple-slit diffraction grating. This will be the case away from synchronism and when the small perturbation condition (58) is valid for the entire multiple-helix interaction. That is, under conditions where the solution of the linear equations (3) and (4) with $\dot{z}$ taken constant is a good approximation. We have found, for example, that the solutions of these equations for the motion through two helices gives an expression for $\dot{x}^2+\dot{y}^2$ similar to that for the intensity produced by a double-slit diffraction grating. It is simply expression (56) multiplied by a factor $4\cos^2\gamma$ where $\gamma = (k\dot{z}-\omega_c)(t_1+t_2)/2$, t_1 being the transit time for each helix, and t_2 the time of flight between the helices. This additional factor gives a fine structure within the $\mathrm{sinc}^2\,\Gamma/2$ envelope.

Although we do not wish at this time to pursue the problem of particle trapping in plasma mirror devices in any depth, it would seem that both the analytic results and the physical model given here are useful in connection with this question. When a helical field is placed between confining magnetic mirrors to increase the magnetic moment of the ionized particles, reflections cause the particles to travel through it in the $+z$ and $-z$ directions alternatively. The sense of rotation of charged particles is unchanged in a mirror reflection, but the velocity $\dot{z}$ and hence the sense of the helical trajectory are reversed. Thus on the return transit in the $-z$ direction the particle and the $\boldsymbol{b}$ field rotate with opposite handedness. For a forward transit the energy transfer is given by (44) or (52), while a reverse transit is described by (59). If multiple transits through the helix occur with completely random phases, as seems likely, we can treat the eventual escape through the mirrors as a random walk problem each 'step' being the randomly directed velocity acquired in a single transit, with escape corresponding to the return of the particle to a small region near the origin of phase space whose size is determined by the loss cone of the mirror. This would be equivalent to the diffusion formulation of Wingerson *et al.* (1964). In the opposite extreme, any component of phase coherence between one transit and the next can be thought of in terms of the well-known curves of multiple slit diffraction theory (Born and Wolff 1959). Of the particles within a range of initial (asynchronous) velocities which are trapped beyond the mirror loss cone following the first transit, a certain fraction associated with the minima of the $\sin^2(\Gamma/2)\cos^2\gamma$ against $\dot{z}$ curve will escape after the second transit. These will be the particles which, by virtue of their times of passage into the mirror and back again, re-negotiate the helix with such phases that their initial circular motion is unwound in the next pass. Subsequent transits would tend to successively sweep other narrow ranges of the velocity spectrum through the loss cone.

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